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Calc 323 11/29/21

§16.7: Surface Integrals

Last time: $\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v)) |\vec{S}_u \times \vec{S}_v| dA$

where $\vec{S}(u,v)$ parameterizes S on domain D.

Ex: Compute $\iint_S x^2\,dS$ for S, the unit sphere @ origin.

Sol: First, we parameterize the unit sphere by

$$\vec{S}(\theta,\phi) = \langle \sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi) \rangle$$

spherical coords

on $(\theta,\phi) \in [0,2\pi]\times[0,\pi]$

$$\vec{S}_\theta = \langle -\sin(\phi)\sin(\theta), \sin(\phi)\cos(\theta), 0 \rangle$$
$$= \sin(\phi)\langle -\sin(\theta), \cos(\theta), 0 \rangle$$

$$\vec{S}_\phi = \langle \cos(\phi)\cos(\theta), \cos(\phi)\sin(\theta), -\sin(\phi) \rangle$$

$$\therefore \vec{S}_\theta \times \vec{S}_\phi = \sin(\phi) \begin{vmatrix} \hat{i} & \hat{j} & k \\ -\sin\theta & \cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix}$$

$$= \sin(\phi)\langle -\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), -\cos(\phi)\sin^2(\theta) - \cos(\phi)\cos^2(\theta) \rangle$$

$$= \sin(\phi)\langle -\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), -\cos(\phi) \rangle$$

$$\therefore \iint_S x^2\,dS = \iint_D \sin^2(\phi)\cos^2(\theta) |\sin(\phi)\langle -\sin(\phi)\cos(\theta), -\sin(\phi)\sin(\theta), -\cos(\phi)\rangle| dA$$

$$= \iint_D \sin^3(\phi)\cos^2(\theta)\sqrt{\sin^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) + \cos^2(\phi)}\,dA$$

$$= \iint_D \sin^3(\phi)\cos^2(\theta)\,dA =$$

→

$$= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \sin(\phi)(1-\cos^2(\phi)) \cdot \tfrac{1}{2}(1+\cos(2\theta))\, d\phi\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} (1+\cos(2\theta)) \int_{u=1}^{-1} -(1-u^2)\, du\, d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{2\pi} (1+\cos(2\theta)) \left[u - \tfrac{1}{3}u^3\right]_{u=1}^{-1} d\theta$$

$$= -\frac{1}{2} \int_{\theta=0}^{2\pi} (1+\cos(2\theta))\left(\left(-1+\tfrac{1}{3}\right)-\left(1-\tfrac{1}{3}\right)\right) d\theta$$

$$= \frac{1}{2} \cdot \frac{4}{3} \int_{\theta=0}^{2\pi} (1+\cos(2\theta))\, d\theta$$

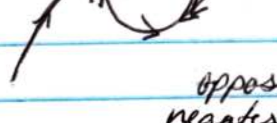
$$= \frac{2}{3}\left[\theta + \tfrac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} = \frac{2}{3}(2\pi - 0) = \boxed{\frac{4}{3}\pi.}$$

GOAL: Build a theory of surface integrals for vector fields (analogous to line integrals)

BUT: We need to think about "orientation" for surfaces first...

("direction")
↳ should involve the right-hand rule.–

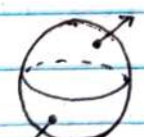
For line integrals:



opposite direction negates the line integral result...

A: Orientation means (for surfaces) "consistent choice of normal to the tangent planes".

Ex:



pos. orientation: OUTWARD. ∴ neg. orientation: INWARD.

alternatively, we want "counterclockwise from above" orientation to be positive

pos. orientation on torus is open outward.

Q: Can we do this for every surface?

Choose consistent pos. orientation.
Mobius strip $\rightarrow$ non-orientable...

NB: Our theory of surface integrals chokes on non-orientable surfaces... From here on, the surfaces we work with are orientable.

NB2: Choosing a parameterization of S by $\vec{S}(u,v)$ automatically chooses an orientation: $\vec{n}(u,v) = \dfrac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}$

Defn: The flux of vector field $\vec{v}$ across surface S is $\iint_S \vec{v} \cdot d\vec{S} = \iint_S \vec{v} \cdot \vec{n}\, dS$

$$= \iint_D \vec{v}(u,v) \cdot \frac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}\, dA$$

$$= \iint_D \vec{v} \cdot (\vec{S}_u \times \vec{S}_v)\, dA$$

where $\vec{S}(u,v)$ is a parameterization of S on domain D.

Ex: Compute the flux of $\vec{v} = \langle z, y, x \rangle$ across the sphere of radius 1, centered @ origin.

~~Sol:~~ Convention: if orientation not explicitly given, it is implicitly the positive orientation.

Sol: As before, $\vec{S}(\theta, \phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$ on $D = [0, 2\pi] \times [0, \pi]$ and

$$\vec{S}_\theta \times \vec{S}_\phi = -\sin\phi \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

Check: outward orientation?

At a test point, the ~~north~~ east pole, ~~(0,0,1)~~ (1,0,0)

i.e. $(\theta,\phi)=(0,\frac{\pi}{2})$

$$\therefore (\vec{S}_\theta \times \vec{S}_\phi)(0,\tfrac{\pi}{2}) = -\langle 1,0,0\rangle = \langle -1,0,0\rangle$$

(inward orientation)

$\therefore$ we need to use $-(\vec{S}_\theta \times \vec{S}_\phi)$ instead.

now, $\vec{v}(\theta,\phi)\cdot(\vec{S}_\theta,\vec{S}_\phi)$

$$=\langle \cos\phi,\ \sin\phi\sin\theta,\ \sin\phi\cos\theta\rangle$$

$$\cdot -\sin\phi\langle \sin\phi\cos\cancel{\phi}\theta,\ \sin\phi\sin\theta,\ \cos\phi\rangle$$

$$= -\sin\phi\,(2\cos\phi\sin\phi\cos\theta + \sin^2\phi\sin^2\theta) =$$

$$\therefore \iint_S \vec{v}\cdot d\vec{S} = \iint_D \vec{v}\cdot(-\vec{S}_\phi \times \vec{S}_\theta)\,dA$$

$$= \iint_D \sin(\phi)(2\cos\phi\sin\phi\cos\theta + \sin^2\phi\sin^2\theta)\,dA$$

$$= 2\iint_D \cos\phi\sin^2\phi\cos\theta\,dA + \iint_D \sin^3\phi\sin^2\theta\,dA$$

Now $\iint_D \cos\phi\sin^2\phi\cos\theta\,dA$

$$= \int_{\phi=0}^{\pi} \cos\phi\sin^2\phi \int_{\theta=0}^{2\pi}\cos\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{\pi} \cos\phi\sin^3\phi\cdot 0\,d\phi = \boxed{0}$$